Stochastic Stability of a Satellite Influenced by Aerodynamic and Gravity Gradient Torques

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Quantitative as well as qualitative results dealing with stochastic stability of a satellite influenced by aerodynamic and gravity gradient torques are presented. In particular, parameter regions denoting that the solution process is "stable with probability one" are found, along with estimates of the probability of convergence to equilibrium positions. The atmospheric density uncertainty introduces the randomness into the system. Stochastic Liapunov functions are used to obtain the stability results, and probability estimates of convergence to equilibrium positions are found by applying supermartingale inequalities. As a prelude of the satellite application, a heuristic explanation of the theory of stochastic Liapunov functions and the techniques of applying this theory to design problems are presented. A complete treatment of the satellite application with its impact on design considerations is provided.

1. Introduction

THE problem of investigating the characteristics of motion of a satellite subjected to gravity gradient and aerodynamic torques has occupied the attention of engineering and scientific literature for the past ten years. Schrello[1] employs Floquet theory to investigate Lagrange stability of the satellite system; Magiros and Dennison[2] apply Liapunov theory to the solution of this problem. Various other authors, Moran,[3] Beletskii,[4] Baker,[5] Debra,[6] and Roberson[7] for example, treated similar problems in the early part of the last decade. More recently, in 1969, Frick[8] employed Liapunov's direct method to study the problem, and Modi and Brereton[9,10] investigated periodicity in two papers. The aerodynamic density has a significant influence on the applied moments and generally is not accurately known. The need for a systematic approach to considering this inaccuracy is imperative, since the references mentioned are deterministic in their approach.

The atmospheric density is modeled as a stochastic process, and the equations of motion are expressed as a set of Ito stochastic differential equations, along with more basic and theoretical developments in the works of Sheporaitis,[11] and Ku and Sheporaitis.[12] In these papers, Kushner's[13] theory of stochastic stability in Euclidean space is extended to cylindrical state spaces, with the object of studying global properties of physical systems with uncertainty, having angular variables. The satellite system is developed as a diffusion process of a cylinder, and stochastic Liapunov theory is applied. This paper extends these works[11,12] by obtaining more detailed information concerning parameter regions for stability and sharper estimates of the probability of convergence to equilibrium positions. In addition, a sketch of the theory of stochastic Ito equations, and the theory and application of stochastic Liapunov functions are presented in section 2 as a prelude to the satellite application.

2. Preliminaries and Background

The applications of stochastic differential equations and stochastic stability are not widespread in engineering literature. For this reason, a heuristic explanation of these concepts is presented here; for more details see Kushner,[13] Doob,[14] and Dynkin.[13-15]

A Markov process that frequently occurs in engineering applications is one that is motivated by

$$\dot{x} = m(x) + \sigma(x)\dot{z} \tag{1}$$

where x takes values in R^n, and $\dot{z}$ is the familiar "Gaussian white noise" vector. The vector $m(x) = [m_1(x), \ldots, m_n(x)]$ and the matrix $\sigma(x) = [\sigma_{ij}(x)]$ satisfy a regularity condition, a boundedness condition, and a uniform Lipshitz condition. System 1, however, is purely formal since almost all sample functions of the process are nondifferentiable even though continuous. System 1 is formulated more precisely in terms of the Ito stochastic integral equation,

$$x_t = x_0 + \int_0^t m(x_s)ds + \int_0^t \sigma(x_s)dz(s) \tag{2}$$

where x_t is written for $x(t)$ and $z(t)$ is a vector Wiener process with independent increments satisfying† $E\{[z(t) - z(s)][z(t) - z(s)]^T\} = I|t - s|$. The second integral of Eq. (2) is interpreted as a stochastic integral; the essential difference between the definition of a stochastic integral and that of a Riemann or Stieltjes integral is that limits of partial sums are replaced by limits in the mean of appropriate random variables. The process x_t defined by Eq. (2) is well defined; solutions can be shown to exist by the technique of successive approximations and are unique up to stochastic equivalence. The process x_t can be thought of undergoing in time interval $(t, t + dt)$ a deterministic displacement $m(x_t)dt$ and a random displacement $\sigma(x_t)dz(t)$, where the random vector is Gaussian with mean 0 and covariance matrix $\sigma(x_t)\sigma^T(x_t)dt$. For cylindrical phase spaces, in order to study global properties of physical systems having angular variables, the interpretation of Eq. (2) is altered slightly[11,12]; the process is called a diffusion process on a cylinder.

A powerful technique for analyzing the qualitative behavior of these systems is that of stochastic stability and stochastic Liapunov functions. Two properties are of interest—the local property of "stability with probability one," and the global property of "stability with probability p with respect to G." Essentially, a process is stable with probability one if the probability that the process leaves a given neighborhood of the origin can be made arbitrarily small if the initial perturbation is small enough. A process is stable with probabil-

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† E and E_x denote expectation and conditional expectation, respectively; I denotes the identity matrix; and B^T is the transpose of the matrix B.

ity p with respect to G if it is stable with probability one, and if the probability that the process converges to the origin is greater than p for any initial state in G.

The concept of a Liapunov function is a powerful tool in the study of stability of deterministic systems. Briefly, a Liapunov function $V(x)$ is a generalized energy function that behaves as the system state, i.e., convergence of the function to zero implies the system state converges to the equilibrium point of interest. Then, if $\dot{V} \leq 0$ on all trajectories, the fundamental theorem of calculus

$$V(x_t) = V(x_0) + \int_0^t \dot{V}(x_s)ds \leq V(x_0) \qquad (3)$$

implies the stability properties of interest; for example, Eq. (3) implies that given any neighborhood U about the equilibrium position, all initial states chosen from a sufficiently small neighborhood yield trajectories that never leave U. Of course, the smoothness properties of the function V and the trajectories x_t are taken for granted and are usually no problem for physical systems with no uncertainty. However, the stochastic System 2, as was mentioned earlier, does not have the necessary smoothness properties. In addition, requiring that all trajectories convergence to the equilibrium position in a stochastic system is an unrealistic requirement for stability. Hence, a stochastic analogue of differentiation, which can introduce an averaging procedure, is needed to construct a theory of stochastic Liapunov functions. There are several such operators that satisfy the aforementioned requirements. Subtle differences exist between them which deal primarily with the classes of functions on which they act; in addition, some differences arise only when they are applied to general Markov processes. Of interest here are the weak infinitesimal operator $\tilde{A}$ and the differential generator $\mathfrak{L}$. $\tilde{A}$ is defined by

$$\tilde{A}f = w\text{—}\lim_{\delta \downarrow 0} [E_x f(x_\delta) - f(x)]/\delta$$

where w-lim is the weak or pointwise limit as opposed to a uniform limit. The class of functions on which $\tilde{A}$ acts are those for which the limit exists and which satisfy a specific regularity condition. One can view $\tilde{A}f$ as the average derivative of f. The differential generator $\mathfrak{L}$ is a second-order differential operator that pertains to Markov processes of the type considered here. It provides a means for computing $\tilde{A}f$ since $\mathfrak{L}f = \tilde{A}f$ for f, a twice continuously differentiable function. For System 2, $\mathfrak{L}$ is defined by

$$\mathfrak{L}f = \sum_{i=1}^{n} m_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} S_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}$$

where $[S_{ij}(x)] = \sigma(x)\sigma^T(x)$.

The stochastic version of the Liapunov function is essentially the same as in the deterministic case; the difference is that convergence of the Liapunov function to zero in a given probability sense implies convergence of the system state to the equilibrium position in the same sense. Supermartingales play an important role in the theory; a supermartingale process $(x_t, t \geq 0)$ is one that satisfies the inequality $Ex_t \leq Ex_s$ for $s < t$. Then if the stochastic Liapunov function $V(x)$ is constrained to be decreasing on the average along trajectories, that is $\tilde{A}V = \mathfrak{L}V \leq 0$, the stochastic analogue of the fundamental theorem of calculus

$$E_{x_0}[V(x_T)] = V(x_0) + E_{x_0} \int_0^T \tilde{A}V(x_t)dt \leq V(x_0)$$

along with supermartingale inequalities imply the desired results. There are subtle mathematical considerations above those mentioned that must be taken care of in a rigorous treatment; however, the purpose of this section is to explain the basic ideas. A more precise version of the theorem to be applied in the satellite application is the following theorem of Sheporaitis.[11]

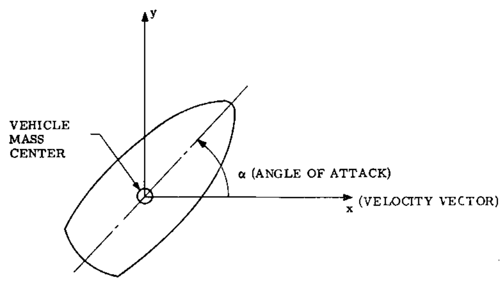

Fig. 1 Satellite pitch plane.

Theorem: Let G be an open set in S^1XR containing the equilibrium state of interest. Let $V(x)$ be a function on G satisfying 1) for‡ $x \epsilon G$, $\beta(||x||) \leq V(x) \leq K||x||^p$ for some $p > 0$, $K > 0$, with β a continuous increasing scalar function with $\beta(0) = 0$, defined on $[0, \infty)$; 2) $V(x)$ is a twice continuously differentiable function on G; and 3) $\mathfrak{L}V(x) \leq -\alpha(x)$ on G, where $\alpha(x)$ is a continuous function on G with $\alpha(x) > 0$ for $x \neq 0$. Then if $\lambda = \sup \{\lambda' : [x : V(x) < \lambda'] \subset G\}$, for any $x_0 \epsilon G$, the process is stable with probability $1 - V(x_0)/\lambda$ with respect to G for λ finite, and with probability 1 with respect to G if λ is not finite.

The construction of Liapunov functions is largely an art with very few systematic techniques available. However, typical forms of the Liapunov function assumed are quadratic functions in the state components plus an integral of the nonlinearity. For more details see the works of Kushner[16–18] and Ku.[19,20]

3. Satellite Influenced by Aerodynamic and Gravity Gradient Torques

The model for a satellite influenced by aerodynamic and gravity gradient torques used in the study of the stochastic stability does not differ significantly from those developed in the literature, with the exception of the introduction of the density uncertainty and possibly a more general aerodynamic moment. The satellite geometry is given in Fig. 1. Schrello[1] shows by means of appropriate references that aerodynamic effects have no first-order effect on rolling motion, and gravitational torques do not significantly influence yawing motion; hence, the motion characteristics will be restricted to a pitch plane analysis. Nicolaides[21] investigates the aerodynamic moments on a vehicle undergoing constrained pitching motion. A static moment, arising from the action of lift forces on the vehicle, is assumed of the form

$$M_s = -(\tfrac{1}{2})\rho v^2 S d_1 C_s g(\alpha) \qquad (4)$$

where ρ is the air density, v is the satellite velocity magnitude, C_s is an aerodynamic coefficient, S is the reference area, and d_1 is the reference length. The function $g(\alpha)$ is defined on $[-\pi,\pi]$ satisfying I) g is continuously differentiable on $[-\pi,\pi]$; II) $g(-x) = -g(x) < 0$ on $(0,\pi)$; III) $g(0) = g(\pi) = 0$; and IV) $|g(\alpha)| \leq |\alpha|$ on $[-\pi,\pi]$. Essentially the aerodynamic static moment is proportional to an odd function of α; a typical example is afforded by sin α. A dynamic moment M_D, which can be considered a damping moment, is assumed to be proportional to $\dot{\alpha}$; more precisely

$$M_D = (-\tfrac{1}{2})\rho v^2 S d_1 C_D \dot{\alpha} \qquad (5)$$

where C_D is a damping coefficient.

Assuming the satellite to be a body of revolution, the torque M_G on the satellite due to a divergent field, namely the inverse square gravitational field, is given by Nidley[22] as

$$M_G = (\tfrac{3}{2})\dot{\omega}^2(I_P - I_R) \sin 2\alpha \qquad (6)$$

‡ For $x = (x_1, x_2) \epsilon S^1XR$, $||x|| = (x_1^2 + x_2^2)^{1/2}$.

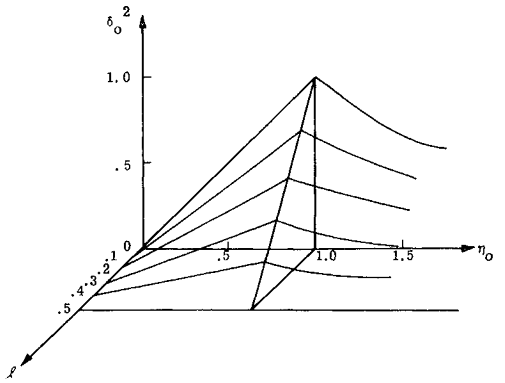

Fig. 2 Parameter region for stochastic stability: δ_0 = density uncertainty ratio, η_0 = normalized damping, and l = gravity to static aerodynamic moment ratio.

Here I_P and I_R are the pitch and roll moments of inertia, and $\dot{\omega}$ is the orbital rate which is constant in a circular orbit.

Treating the quantities v, S, d_1, C_D, c_s, $\dot{\omega}$, I_P and I_R as constants, Newton's laws of motion yield

$$I_P\ddot{\alpha} = -(\tfrac{1}{2})\rho v^2 S d_1[C_s g(\alpha) + C_D\dot{\alpha}] + (\tfrac{3}{2})\dot{\omega}(I_P - I_R)\sin 2\alpha \quad (7)$$

Then Eq. (7) becomes by two successive integrations,

$$\alpha(t) = \alpha(0) + \dot{\alpha}(0)t + (\tfrac{3}{2})\dot{\omega}(I - I_R/I_P)\int_0^t\int_0^\tau \sin 2\alpha(s) \times ds d\tau - \left(\frac{1}{2I_P}\right)v^2 S d_1 \int_0^t\left(\int_0^\tau \rho\{C_s g[\alpha(s)] + C_D\dot{\alpha}(s)\}ds\right)d\tau \quad (8)$$

With the object of introducing the variation of α as an It_o stochastic integral equation let the last integral of Eq. (8) be modeled as§

$$-\left(\frac{1}{2I_P}\right)v^2 S d_1\rho_0\int_0^t \left\{\int_0^\tau C_s g[\alpha(s)]ds + C_D\alpha(\tau)\right\} + \delta\left[\left(\int_0^\tau \{C_s g[\alpha(s)] + C_D\dot{\alpha}(s)\}dz(s)\right)\right]d\tau$$

Here ρ_0 and δ are constants, and $z(s)$ a scalar Wiener process. This assumption can be interpreted as representing the atmospheric density as a stochastic process, such that the integrated sum of the static and dynamic aerodynamic moments is a Markov process with Gaussian increments; the incremental mean and variance are $-\{(\tfrac{1}{2})\rho_0 v^2 S d_1 C_s g(\alpha) + C_D\dot{\alpha}\}\Delta t$ and $\{(\tfrac{1}{2})\rho_0 v^2 S d_1[C_s g(\alpha) + C_D\alpha]\}^2\delta^2\Delta t$, respectively.¶ The parameter δ will be sized in the subsequent stability analysis to establish sufficiency conditions for stability with probability one.

Parameters, which are used in the construction of parameter regions for stability, are defined as follows: 1) $Q = \left(\frac{1}{2I_P}\right)\rho_0 v^2 S d_1 C_s$, the maximum static aerodynamic moment divided by the pitch inertia; 2) $\eta = C_D/C_s$, the ratio of the dynamic to static aerodynamic coefficients or, equivalently, the ratio of the dynamic moment divided by $\dot{\alpha}$ to the static moment divided by $g(\alpha)$; and 3) $l = (\tfrac{3}{2})\dot{\omega}^2(1 - I_R/I_P)/Q$, the ratio of the maximum gravitational torque to the maximum static aerodynamic torque. Then if

$$x_1(t) = \alpha(t), x_2(t) = \dot{\alpha}(0) + Q\eta\alpha_1(0) - Q\int_0^t \{g[\alpha(s)] - l\sin 2\alpha(s)\}ds - Q\delta\int_0^t \{g[\alpha(s)] + \eta\dot{\alpha}(s)\}dz(s)$$

the system defined by the modified version of Eq. (8) assumes a form as in Eq. (2) with

$$m(x) = \begin{bmatrix} x_2 - Q\eta x_1 - \\ Qg(x_1) + Ql\sin 2x_1 \end{bmatrix}$$
$$\sigma(x) = \begin{Bmatrix} 0 & 0 \\ 0 & -Q\delta[x_2\eta - Q\eta^2 x_1 + g(x_1)] \end{Bmatrix} \quad (9)$$

It should be noted that $x = (x_1, x_2)$ takes values in S^1XR. It can be seen that (0,0) and $(\pi, \pi Q\eta)$ are equilibrium positions of the system. Motivated by investigations of deterministic systems, the (0,0) solution is the equilibrium position of interest; in all references to equilibrium positions in the following sections it is understood that (0,0) is meant.

4. Stochastic Stability of a Satellite

With the object of applying the theorem of §2 to determine the stability with probability p with respect to G, where p and G are to be determined, select as a stochastic Liapunov function

$$V(x) = ax_2^2 - bQ\eta x_1 x_2 + cQ^2\eta^2 x_1^2 + dQ\int_0^{x_1} g(y)dy - Ql(1 - \cos 2x_1) \quad (10)$$

$a, b, c, d > 0$. Using Eq. (3) to compute $\mathfrak{L}V$,

$$\mathfrak{L}V(x) = x_2 g(x_1)Q(-2a + d + 2aQ\delta^2\eta) + x_1 g(x_1)Q^2\eta(b - d - 2aQ\eta\delta^2) + aQ^2\delta^2 g^2(x_1) + x_1^2 Q^3\eta^3(-2c + aQ\eta\delta^2) + x_2^2 Q\eta(-b + aQ\eta\delta^2) + x_1x_2Q^2\eta^2(b + 2c - 2aQ\eta\delta^2) + Q^2\eta l x_1\sin 2x_1$$

With the object of satisfying condition 3 of the Theorem, set the coefficient of $x_2 g(x_1)$ equal to zero and set $b = 2c$. Then $d = 2a(1 - Q\eta\delta^2)$ and $\mathfrak{L}V(x)$ becomes

$$\mathfrak{L}V(x) = x_1 g(x_1)Q^2\eta(b - 2a) + aQ^2\delta^2 g(x_1) + Q\eta(-b + aQ\eta\delta^2)(Q\eta x_1 - x_2)^2 + Q\eta l x_1 \sin 2x_1$$

By conditions II and IV on $g(\alpha)$, setting $a = b = 1$,

$$\mathfrak{L}V(x) < -Q\eta^2 x_1[(1 - \delta^2/\eta)g(x_1) - l\sin 2x_1] - Q\eta(1 - Q\eta\delta^2)(Q\eta x_1 - x_2^2) \quad (11)$$

for $(x_1, x_2) \not\subset (0,0) \cup (\pi, \pi Q\eta)$. The Liapunov function now becomes

$$V(x) = \tfrac{1}{2}x_2^2 + \tfrac{1}{2}(x_2 - Q\eta x_1)^2 + 2Q(1 - Q\eta\delta^2)\cdot\int_0^{x_1} g(y)dy - Ql(1 - \cos 2x_1) \quad (12)$$

Now $V(x)$ satisfies conditions 1 and 2 of the Theorem for $x \neq (\pi, \pi Q\eta)$ by virtue of conditions I and IV on $g(\alpha)$, and $\mathfrak{L}V(x)$ satisfies condition 3 by virtue of conditions II and IV

§ It should be noted that the lack of differentiability in a stochastic system can be circumvented by defining $\dot{\alpha}(t)$ by

$$\dot{\alpha}(t) = \dot{\alpha}(0) + (\tfrac{3}{2})\dot{\omega}(1 - I_R/I_Q)\int_0^t \sin 2\alpha(s)d(s) - (\tfrac{1}{2}I_P)V^2 S d_1\rho_0\left(\int_0^t C_s g[\alpha(s)]ds + C_D\alpha(s) + \delta\left\{\int_0^t [C_s g\{\alpha(s)\} + C_D\dot{\alpha}(s)]dz(s)\right\}\right)$$

¶ If one envisions the physical model as the limit of models with additive wide-band noise, the dynamical contribution of the system must be altered by the addition of $Q\eta\delta^2[x_2\eta - Q\eta^2 x_1 + g(x_1)]$ to the second component of $m(x)$ in Eq. (9). For more details see Wing and Zakai.[24] However, the model for this analysis will be taken as aforementioned. The reason for this is twofold: 1) the physical model is not necessarily the limit of wideband process models of the specified type—more investigation is required in this area; and 2) the model chosen follows directly from the associated deterministic system.

on $g(\alpha)$ if

$$\delta^2 < \min[\eta, 1/(Q\eta)] \tag{13}$$

$$(1 - Q\eta\delta^2)|g(x)| \geq l|\sin 2x| \tag{14}$$

$$(1 - \delta^2/\eta)|g(x)| \geq l|\sin 2x| \tag{15}$$

Conditions (13) ensures that the coefficients of**

$$\int_0^{x_1} g(y)dy, \; -Q\eta(Q\eta x_1 - x_2)^2,$$

and $g(x_1)$ are positive. Condition (14) guarantees that

$$2Q(1 - Q\eta\delta^2)\int_0^{x_1} g(y)dy - Ql(1 - \cos 2x_1) = 2Q \times \int_0^{x_1} [(1 - Q\eta\delta^2)g(y) - l\sin 2y]dy$$

is positive for $x_1 \neq 0$, and condition (15) ensures that the term $(1 - \delta^2/\eta)\cdot g(x_1) - l\sin 2x_1$ of $\mathcal{L}V(x)$ is an odd function. Conditions 13–15 represent sufficiency conditions for the local property of stability with probability one; for the global property of stability with probability p with respect to G, the p and G must be determined in addition to the sufficiency conditions (13–15). If the special case $g(x_1) = \sin x_1$ is used, conditions (14) and (15) are replaced by

$$l < (\tfrac{1}{2})\min[(1 - \delta^2/\eta), (1 - Q\eta\delta^2)] \tag{16}$$

If the parameters δ^2 and η are multiplied by the natural frequency $(Q)^{1/2}$ of the associated deterministic aerodynamic system, conditions (13) and (16) can be simplified to

$$\delta_0^2 < \min\{\eta_0, 1/\eta_0\} \tag{17a}$$

$$l < (\tfrac{1}{2})\min\{(1 - \delta_0^2\eta_0), (1 - \eta_0\delta_0^2)\} \tag{17b}$$

where

$$\delta_0^2 = \delta^2(Q)^{1/2} \tag{18a}$$

$$\eta_0 = \eta(Q)^{1/2} \tag{18b}$$

Now δ_0 and l are assumed to be non-negative and η_0 is assumed to be positive. Hence, $\eta_0 \geq 1$ if and only if $1 - \delta_0^2/\eta_0 \geq 1 - \eta_0\delta_0^2$, and thus the second inequality of Eq. (17) becomes

$$\delta_0^2 < \begin{cases} \eta_0(1 - 2l), & \eta_0 < 1 \\ (1 - 2l)/\eta_0, & \eta_0 \geq 1 \end{cases}$$

Thus, the second inequality of Eq. (17) is equivalent to

$$\delta_0^2 < \min\{\eta_0(1 - 2l), (1 - 2l)/\eta_0\} \tag{19}$$

Since $\delta_0^2 \geq 0$, l takes values in $[0,\frac{1}{2}]$; that is, $1 - 2l$ takes values in $[0,1]$. Clearly, inequality of condition (19) implies both inequalities of condition (17). Thus, the parameter region for stability with probability one in the l, η_0, δ_0^2 space as defined by conditions (13–15), or equivalently by condition (19), is that region that lies under the surface defined by replacing the inequality in condition (19) by an equality. This is shown in Fig. 2.

In addition to the qualitative information provided by the stochastic Liapunov theory some conservative quantitative estimates of the probability of convergence to equilibrium positions are furnished. The Liapunov function of Eq. (13) when $g(x_1) = \sin x_1$ becomes

$$V(x) = \tfrac{1}{2}x_2^2 + \tfrac{1}{2}(x_2 - Q\eta x_1)^2 + 4Q(1 - Q\eta\delta^2)\sin^2(x_1/2) - 2Ql\sin^2 x_1 \tag{20}$$

As was mentioned before, any open set G over which the Liapunov function is defined and satisfies properties 1–3 of the Theorem cannot include the point $(x_1,x_2) = (\pi,\pi Q\eta)$.

** It should be noted that the integral over $[0,x_1]$ for any x_1 of an odd function is non-negative.

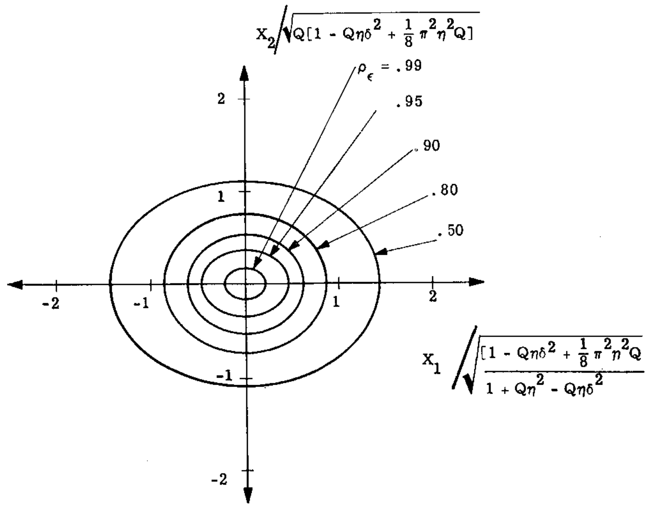

Fig. 3 Normalized phase space with regions denoting convergence to origin with probability p_μ.

The probability that a given sample function, which has x_0 as its initial point in G, will converge to the equilibrium position is $p_0 = 1 - V(x_0)/\lambda$, where λ defines the biggest set of the form $(x:V(x) < \lambda)$ contained in G. Hence, to make p_0 as close to 1 as possible, λ must be made as large as possible. Thus, choose λ to be $V(\pi,\pi Q\eta)$, or

$$\lambda = (\tfrac{1}{2})(\pi Q\eta)^2 + 4Q(1 - Q\eta\delta^2)$$

and define G by $G = \{x:V(x) < \lambda\}$ where $V(x)$ is defined by Eq. (20). Also define $G_\mu = \{x:V(x) < \lambda\mu\}$ where $\mu \epsilon (0,1)$. Thus, for any sample function with initial condition $x_0 \epsilon G_\mu$, the probability p_μ that it will converge to the equilibrium position is $1 - \mu$. By virtue of the analysis in the appendix $(y_2 - Q\eta y_1)^2 \leq 2y_2^2 + 2Q^2\eta^2 y_1^2$; in addition, $\sin y_1/2 \leq y_1/2$ and $-2Ql\sin^2 y_1 \leq 0$. Thus if

$$H_\mu = \{x:(\tfrac{3}{2})x_1^2 + Q(Q\eta^2 + 1 - Q\eta\delta^2)x_1^2 \leq (\tfrac{1}{2})\mu(\pi Q\eta)^2 + \mu 4Q(1 - Q\eta\delta^2)\}$$

then $G\mu \supseteq H\mu$. But

$$H_\mu = \{x:x_1^2/a^2 + x_2^2/b^2 \leq 1\}$$

where

$$a = 2\{\mu[1 - Q\eta\delta^2 + (\tfrac{1}{8})\pi^2\eta^2 Q]/(1 + Q\eta^2 - Q\eta\delta^2)\}^{1/2}$$
$$b = \{(\tfrac{8}{3})\mu Q[1 - Q\eta\delta^2 + (\tfrac{1}{8})\pi^2\eta^2 Q]\}^{1/2}$$

H_μ is an ellipse, the biggest contained in G_μ subject to the inequality estimates made previously, such that all trajectories starting within it converge to the equilibrium position with at least probability $p_\mu = 1 - \mu$. This is shown in Fig. 3, where x_1 and x_2 are normalized with respect to the factors $\{[1 - Q\eta\delta^2 + (\frac{1}{8})\pi^2\eta^2 Q]/(1 + Q\eta^2 - Q\eta\delta^2)\}^{1/2}$ and $\{Q[1 - Q\eta\delta^2 + (\frac{1}{8})\pi^2\eta^2 Q]\}^{1/2}$, respectively. In many cases, in particular those were Q, η, and δ are dominated by unity, these factors become 1 and $Q^{1/2}$, respectively.

5. Conclusions

The application of stochastic Liapunov theory can be seen to be a powerful tool to obtain information about physical systems with uncertainty. Given the mathematical model of the physical phenomena, the analysis is exact with no approximations. Useful design data is provided; in this case, the parameter regions for stochastic stability and phase space regions for convergence to equilibrium positions are furnished. Of course, as in the case with any technique, care must be exercised in its application and the interpretation of the results. One characteristic feature of Liapunov theory is its conservatism; this is evident since the Theorem states sufficiency but not necessary conditions for stability. There is no assurance for example, that the parameter region of condition (19) is the largest such region that guarantees stochastic stability. It should be noted from Fig. 2 that the gravity gradient torque introduces instability, with $l = \frac{1}{2}$ being the maximum value sufficient for stability with probability one. Also indicated is that there is an optimum amount of damping, $\eta = 1/Q^{1/2}$ where $Q^{1/2}$ is the natural frequency of the associated deterministic aerodynamic system, which allows the highest noise level and still guarantees stochastic stability. However, it is the author's experience that most vehicles will have such characteristics that will place them in Fig. 2 considerably to the left of $\eta_0 = 1$. For example, a typical vehicle in a 100 nm circular orbit would have $\eta_0 = 0.001$ and $l = 0.015$ thereby allowing $\delta_0 = 0.031$. Only high-damping vehicles, that is those with $C_D \geq 0.01$ sec, will guarantee generous regions of stability. As the altitude of the orbit increases to about 160 nm, l approaches the critical value $\frac{1}{2}$. The appropriateness of stability with probability one as a design criterion is largely dependent on whether or not local probabilistic behavior is important in the physical problem.

The probability estimates of convergence of the equilibrium position show that the size of the convergence region in the phase plane, that is the major and minor axes of the ellipse denoting convergence with probability $1 - \mu$, is proportional to $(\mu)^{1/2}$. Also, the axis in the x_2 direction is proportional to $(Q)^{1/2}$. Other than these relations, there is no strong dependence on the parameters, since these parameters are generally dominated by unity. This effect is caused by the conservatism of the stochastic Liapunov theory and the estimates in obtaining the ellipse H_μ interior to G_μ. For example, the ellipse H_μ is properly contained in G_μ since the inequality $(x_2 - Q\eta x_1)^2 \leq 2x_2^2 + 2Q^2\eta^2x_1^2$ is valid over R^2, whereas (x_1,x_2) take values in the subset $[-\pi,\pi]R$. The probability estimates of convergence to the equilibrium position are nevertheless useful in design situations where this behavior must be predicted with a high degree of certainty. For example, for small Q and η Fig. 3 shows that 11.5° and 36° amplitudes of α are the greatest allowable to guarantee convergence to the equilibrium position with probabilities 0.99 and 0.90, respectively.

Further study is warranted to investigate other possible sources of uncertainty. Errors in vehicle characteristics such as C_s, C_D, I_P, and I_R will affect the aerodynamic and gravitational moments. Furthermore uncertainties in trajectory parameters v and $\dot{\omega}$ will affect these moments. Modeling these quantities as Ito processes similar to the development of section 3 followed with a stochastic Liapunov analysis would be one possible way to attack the problem.

6. Appendix: Inequality Estimates

Here the proof of

$$(x_2 - Q\eta x_1)^2 \leq 2x_2^2 + 2Q^2\eta^2x_1^2 \tag{21}$$

is presented. Suppose that

$$(x + K_1y)^2 \leq K_2x^2 + K_3y^2 \tag{22}$$

Then expanding and collecting terms, Inequality (22) becomes

$$x^2(K_2 - 1) - 2K_1xy + (K_3 - K_1^2)y^2 \geq 0$$

alternately Inequality (22) can be written

$$(xy)A\begin{pmatrix} x \\ y \end{pmatrix} \geq 0$$

where

$$A = \begin{pmatrix} K_2 - 1 & -K_1 \\ -1K_1 & K_3 - K_1^2 \end{pmatrix}$$

By Silvester's criterion for positive semidefiniteness of a quadratic form,[23] necessary and sufficient conditions for Inequality (22) to hold are given by $K_2 \geq 1$, $K_3 \geq K_1^2$, and det $A = 0$. The last condition is equivalent to $K_3(1 - 1/K_2) = K_1^2$. Inequality (21) follows immediately upon selection of $K_1 = -Q\eta$, $K_2 = 2$, and $K_3 = 2Q^2\eta^2$.

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